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VARIATIONAL CONVERGENCE OVER \mathbb{R}^n -UNIFORMLY CONVEX SPACES (Geometric Aspect of Partial Differential Equations and Conservation Laws)

AUTHOR(S):

KUWAE, KAZUHIRO

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VARIATIONAL CONVERGENCE OVER p -UNIFORMLY CONVEX SPACES

KAZUHIRO KUWAE

ABSTRACT. We establish a variational convergence over p -uniformly convex spaces for $p \geq 2$. Variational convergence for Cheeger type energy functionals over L^p -maps into p -uniformly convex space having NPC property of Busemann type and the existence of p -harmonic map for Cheeger type energy functionals with Dirichlet boundary condition are also presented.

1. INTRODUCTION AND MAIN RESULT

This article is a summary of a part of the paper [17] under preparation. We study a variational convergences over p -uniformly convex spaces having NPC property in the sense of Busemann, where a p -uniformly convex space is a natural generalization of p -uniformly convex Banach space. Typical examples of p -uniformly convex spaces are L^p -spaces with $p \geq 2$, CAT(0)-spaces, more concretely, Hadamard manifolds and trees, and so on. If the target space is a p -uniformly convex space having NPC property in the sense of Busemann, then the L^p -mapping space is also a p -uniformly convex geodesic spaces having NPC property in the sense of Busemann, and an energy functional defined in a suitable way becomes convex and lower semi-continuous. Thus, it is reasonable to consider that (H_i, d_{H_i}) and (H, d_H) are all p -uniformly convex geodesic spaces having the weak L -convexity of Busemann type instead of such L^p -mapping spaces (see Definition 3.1 below for the weak L -convexity), and $E_i : H_i \rightarrow [0, \infty]$ and $E : H \rightarrow [0, \infty]$ are convex lower semi-continuous functions with $E_i, E \not\equiv +\infty$. For any $\lambda \geq 0$ and $u \in H$, there exists a unique minimizer, say $J_\lambda^E(u) \in H$, of $v \mapsto \lambda^{p-1}E(v) + d_H^p(u, v)$. This defines a map $J_\lambda^E : H \rightarrow H$, called the *resolvent of E* (see Theorem 5.2 below and [9, 22, 20] for the case $p = 2$). The minimum $E^\lambda(u) := \min_{v \in H} (\lambda^{p-1}E(v) + d_H^p(u, v))$ is called the *Moreau-Yosida approximation* or the *Hopf-Lax formula*. Note that if X is a Hilbert space and if E is a closed densely defined symmetric

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quadratic form on X , then we have $J_\lambda^E = (I + \lambda A)^{-1}$, where A is the infinitesimal generator associated with E . The one-parameter family $[0, +\infty[\ni \lambda \mapsto J_\lambda^E(u)$ gives a deformation of a given map $u \in H$ to a minimizer of E (or a harmonic map), $\lim_{\lambda \rightarrow +\infty} J_\lambda^E(u)$ (if any). Jost [13] studied convergence of resolvents and Moreau-Yosida approximations. Although his study is only on a fixed CAT(0)-space, we extend it for a sequence of p -uniformly convex geodesic spaces having the weak L -convexity of Busemann type with an asymptotic relation (Theorem 5.11 below). This is new even on a fixed p -uniformly convex geodesic spaces having the weak L -convexity of Busemann type.

We can apply our result in the following way. Let $(X_i, q_i) \rightarrow (X, q)$ and $(Y_i, o_i) \rightarrow (Y, o)$ ($i = 1, 2, 3, \dots$) be two pointed Gromov-Hausdorff convergent sequences of proper metric spaces, where ‘proper’ means that any bounded subset is relatively compact, and let us give a positive Radon measure m_i on X_i with full support which converge to a positive Radon measure m on X (see the definition for the convergence of measures in [20]). We are interested in the convergence and asymptotic behavior of maps $u_i : X_i \rightarrow Y_i$ and also energy functionals E_i defined on a family of maps from $X_i \rightarrow Y_i$. We set $L_i^p := L_{o_i}^p(X_i, Y_i, m_i)$ and $L^p := L_o^p(X, Y, m)$. For $u_i, v_i \in L_i^p$ (resp. $u, v \in L^p$), we set $d_{L_i^p}(u_i, v_i) := \|d_{Y_i}(u_i, v_i)\|_{L_i^p}$ (resp. $d_{L^p}(u, v) := \|d_Y(u, v)\|_{L^p}$), where $\|\cdot\|_{L_i^p}$ (resp. $\|\cdot\|_{L^p}$) is the L^p -norm with respect to the measure m_i (resp. m). Consider

$$\mathcal{L}^p := \bigsqcup_i L_i^p \sqcup L^p$$

and endowed the L^p -topology defined in [20] with \mathcal{L}^p . The L^p -topology on \mathcal{L}^p has some nice properties involving the L^p -metric structure of L_i^p and L^p , such as, if $L_i^p \ni u_i, v_i \rightarrow u, v \in L^p$ respectively in L^p , then $d_{L_i^p}(u_i, v_i) \rightarrow d_{L^p}(u, v)$. By their properties we present a set of axioms for a topology on \mathcal{L}^p for $(L_i^p, d_{L_i^p})$ and (L^p, d_{L^p}) . We call such a topology satisfying the axioms the *asymptotic relation between $\{L_i^p\}$ and L^p* (see Definition 4.3). Since L_i^p and L^p are typically improper, the asymptotic relation can be thought as a non-uniform variant of Gromov-Hausdorff convergence.

We now assume that Y_i and Y are p -uniformly convex spaces with common parameter $k \in]0, 2]$ having NPC in the sense of Busemann and satisfying (B) and (C). Then L_i^p and L^p are so. Let E_i (resp. E) be Cheeger type p -energy functional on $H^{1,p}(X_i, Y_i; m_i) (\subset L_i^p)$ (resp. $H^{1,p}(X, Y; m) (\subset L^p)$). Here $H^{1,p}(X_i, Y_i; m_i)$ (resp. $H^{1,p}(X, Y; m)$) is the Cheeger-type p -Sobolev space for L^p -maps with respect to m_i (resp. m) from X_i to Y_i (resp. X to Y) (see Section 6 below). Then E_i (resp. E) is a lower semi-continuous convex functional on L_i^p (resp. L^p). As a corollary of Theorem 5.11 below, we have the following:

Theorem 1.1. *If E_i converges to E in the Mosco sense, then for any $\lambda > 0$ we have the following (1) and (2).*

- (1) E_i^λ strongly converges to E^λ .
- (2) $J_\lambda^{E_i}$ strongly converges to J_λ^E .

Under a suitable condition like uniform Ricci lower bound condition for X_i, X , we can expect that the Mosco convergence of $\{E_i\}$ to E holds. At present, we are still in progress to deduce it.

As an addendum, we also show the existence of p -harmonic map for Cheeger type energy functionals over L^p -maps into p -uniformly convex space having NPC in the sense of Busemann with Dirichlet boundary condition (see Theorem 6.20 below).

2. p -UNIFORMLY CONVEX SPACES

Definition 2.1 (Geodesics). Let (Y, d) be a metric space. A map $\gamma : I \rightarrow Y$ is said to be a *curve* if it is continuous, where $I = [a, b] \subset \mathbb{R}$ is a closed interval. The length $L(\gamma)$ of a curve $\gamma : I \rightarrow Y$ is defined to be

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) \mid a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \right\}.$$

A curve $\gamma : I \rightarrow Y$ is said to be a *minimal geodesic* if $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds for any $s, t \in I, s < t$, equivalently $d(\gamma_r, \gamma_t) = d(\gamma_r, \gamma_s) + d(\gamma_s, \gamma_t)$ for any $r < s < t$. A curve $\gamma : I \rightarrow Y$ is said to be a *geodesic* if for any $s, t \in I, s < t$ with sufficiently small $|t - s|$, $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds. A metric space (Y, d) is called a *R -geodesic space* for $R \in]0, \infty]$ if any two points in Y whose distance is strictly less than R can be joined by a minimal geodesic. We simply say that (Y, d) is a *geodesic space* if it is an ∞ -geodesic space. Throughout this paper, for given $x, y \in Y$, denote by $\gamma_{xy} : [0, 1] \rightarrow Y$ a minimal geodesics from $x =: \gamma_{xy}(0)$ to $y =: \gamma_{xy}(1)$ provided (Y, d) is an R -geodesic space and $d(x, y) < R$.

For $n \in \mathbb{N}$, we denote by $\mathbb{M}^n(\kappa)$ the n -dimensional space form of constant curvature $\kappa \in \mathbb{R}$. Let R_κ be the diameter of $\mathbb{M}^n(\kappa)$, that is, $R_\kappa := \infty$ if $\kappa \leq 0$ and $R_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Definition 2.2 (CAT(κ)-Inequality, see [2]). Let (Y, d) be a metric space and \triangle a geodesic triangle in Y with perimeter strictly less than $2R_\kappa$. Let $\tilde{\triangle}$ be a comparison triangle of \triangle in $\mathbb{M}^2(\kappa)$. We say that \triangle satisfies *CAT(κ)-inequality* if all $p, q \in \triangle$ and its corresponding points $\tilde{p}, \tilde{q} \in \tilde{\triangle}$ satisfy

$$d(p, q) \leq d(\tilde{p}, \tilde{q}).$$

Definition 2.3 (CAT(κ)-Space, see [2]). A metric space (Y, d) is said to be a *CAT(κ)-space* if (Y, d) is a R_κ -geodesic space and all geodesic triangles in Y with perimeter strictly less than $2R_\kappa$ satisfy CAT(κ)-inequality.

Definition 2.4 (p -Uniformly Convex Geodesic Space; cf. Naor-Silberman [25]). A metric space (Y, d) is called p -uniformly convex with parameter $k > 0$ if (Y, d) is a geodesic space and for any three points $x, y, z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0,1]}$ in Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$,

$$(2.1) \quad d^p(z, \gamma_t) \leq (1-t)d^p(z, x) + td^p(z, y) - \frac{k}{2}t(1-t)d^p(x, y).$$

By definition, putting $z = \gamma_t$, we see $k \in]0, 2]$ and $p \in [2, \infty[$. The inequality (2.1) yields the (strict) convexity of $Y \ni x \mapsto d^p(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a p -uniformly convex space is again a p -uniformly convex space with the same parameter. Any L^p space over a measurable space is p -uniformly convex with parameter $k = \frac{8}{4^p p^2} \left(\frac{p-1}{p}\right)^{p-1}$ provided $p \geq 2$. Every CAT(0)-space is a p -uniformly convex space with parameter $k = \frac{8}{4^p p^2} \left(\frac{p-1}{p}\right)^{p-1}$ for $p > 2$ (we can take $k = 2$ if $p = 2$), because \mathbb{R}^2 is isometrically embedded into $L^p([0, 1])$ for $p > 1$ (see [5], [25]) and any L^p -space is p -uniformly convex for $p \geq 2$. Ohta [28] proved that for $\kappa > 0$ any CAT(κ)-space Y with $\text{diam}(Y) < R_\kappa/2$ is a 2-uniformly convex space with parameter $\{(\pi - 2\sqrt{\kappa}\varepsilon) \tan \sqrt{\kappa}\varepsilon\}$ for any $\varepsilon \in]0, R_\kappa/2 - \text{diam}(Y)]$.

Remark 2.5. A Banach space $(Y, \|\cdot\|)$ is said to be *uniformly convex* if

$$\delta_Y(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid x, y \in Y, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon \right\},$$

the modulus of convexity of Y , satisfies $\delta_Y(\varepsilon) > 0$ for $\varepsilon \in]0, 2]$. For $p \geq 2$, $(Y, \|\cdot\|)$ is said to be p -uniformly convex if there exists $c > 0$ such that $\delta_Y(\varepsilon) \geq c\varepsilon^p$ for $\varepsilon \in]0, 2]$. It is known that for $p \geq 2$, $\delta_{L^p}(\varepsilon) = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}} \geq \frac{1}{2^p} \varepsilon^p$ for $\varepsilon \in]0, 2]$. By Lemma 2.1 in [29], if a Banach space $(Y, \|\cdot\|)$ is p -uniformly convex for $p \geq 2$, then there exists $d = d(c, p) > 0$ such that

$$\|(1-t)x + ty\|^p \leq (1-t)\|x\|^p + t\|y\|^p - d\{t(1-t)^p + t^p(1-t)\}\|x-y\|^p$$

for all $x, y \in Y$ and $t \in]0, 1[$. Actually, we can take $d = \frac{c}{p} \left(\frac{p-1}{p}\right)^{p-1}$ as an optimal value. Since $\frac{4}{2^p} \leq (1-t)^{p-1} + t^{p-1} \leq 1$ for all $t \in [0, 1]$, p -uniform convexity of the Banach space implies the p -uniform convexity of geodesic space.

The following propositions can be proved in the same way as in [28]. So we omit its proof.

Proposition 2.6 (cf. Lemma 2.3 in [28]). *Let (Y, d) be a p -uniformly convex space. For $x, y, z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0,1]}$ in*

Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$, we have

$$(2.2) \quad d^p(z, \gamma_t) \leq \frac{2}{k} \cdot \frac{1}{t^{p-1} + (1-t)^{p-1}} \\ \times \left((1-t)^{p-1} d^p(z, x) + t^{p-1} d^p(z, y) - (1-t)^{p-1} t^{p-1} d^p(x, y) \right).$$

Proposition 2.7 (cf. Lemma 2.2 and Proposition 2.4 in [28]). *Any two points in a p -uniformly convex space can be connected by a unique minimal geodesic and contractible.*

Lemma 2.8 (Projection Map to Convex Set). *Let (Y, d) be a complete p -uniformly convex space with parameter $k \in]0, 2]$. The the following hold:*

- (1) *Let F be a closed convex subset of (Y, d) . Then for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d(x, F) = d(\pi_F(x), x)$ holds. We call $\pi_F : Y \rightarrow F$ the projection map to F .*
- (2) *Let F be as above. Then π_F satisfies*

$$(2.3) \quad d^p(z, \pi_F(z)) + \frac{k}{2} d^p(\pi_F(z), w) \leq d^p(z, w), \quad \text{for } z \in Y, w \in F, \\ \text{in particular, } \left(\frac{k}{2}\right)^{1/p} d(\pi_F(z), w) \leq d(z, w) \text{ for } z \in Y, w \in F.$$

Definition 2.9 (Vertical Geodesics). Let (Y, d) be a geodesic space. Take a geodesic η with a point p_0 on it and another geodesic γ through p_0 . We say that γ is *vertical to η at p_0* (write $\gamma \perp_{p_0} \eta$ in short) if for any $x \in \gamma$ and $y \in \eta$,

$$d(x, p_0) \leq d(x, y)$$

holds.

Let (Y, d) be a complete p -uniformly convex space with parameter $k \in]0, 2]$. We consider the following conditions:

- (A) For any closed convex set F in (Y, d) , the projection map $\pi_F : Y \rightarrow F$ satisfies $d(\pi_F(x), y) \leq d(x, y)$ for $x \in Y$, $y \in F$.
- (B) Let γ and η be minimal geodesics among two points such that γ intersects η at p_0 . Then $\gamma \perp_{p_0} \eta$ implies $\eta \perp_{p_0} \gamma$.
- (C) Let σ and η be minimal geodesics among two points such that σ intersects η at p_0 and $\sigma \neq \{p_0\}$. Suppose that γ is a minimal geodesic among two points which contains σ . Then $\sigma \perp_{p_0} \eta$ implies $\gamma \perp_{p_0} \eta$.

Lemma 2.10. (B) implies (A).

Remark 2.11. Theorem 2.13 below shows that the conditions (A), (B), (C) are satisfied for any complete $\text{CAT}(\kappa)$ -space with diameter strictly less than $R_\kappa/2$. For any complete p -uniformly convex space (Y, d) with parameter $k \in]0, 2]$ which is also a weakly L -convex space in the sense

of Busemann for some (L_1, L_2) satisfying the conditions (A), (B), (C), the space $L_h^p(X, Y; m)$ of L^p -maps from (X, \mathcal{X}, m) into Y with a map $h : X \rightarrow Y$ is also a complete p -uniformly convex space with the same parameter $k \in]0, 2]$ which is also a weakly L -convex space in the sense of Busemann for the same (L_1, L_2) . and $L_h^p(X, Y; m)$ satisfies the conditions (A), (B), (C).

Lemma 2.12. Take a geodesic triangle $\triangle ABC$ in $\mathbb{M}^n(\kappa)$ and set $a := d_{\mathbb{M}^n(\kappa)}(B, C)$, $b := d_{\mathbb{M}^n(\kappa)}(C, A)$, $c := d_{\mathbb{M}^n(\kappa)}(A, B)$. Assume $a, b, c < R_\kappa/2$ and $\angle BAC \geq \pi/2$. Then for any point P on AB , $d_{\mathbb{M}^n(\kappa)}(C, A) \leq d_{\mathbb{M}^n(\kappa)}(C, P) \leq d_{\mathbb{M}^n(\kappa)}(C, B)$ holds.

Theorem 2.13. Let $\kappa \in \mathbb{R}$. Any $\text{CAT}(\kappa)$ -space (Y, d) with $\text{diam}(Y) < R_\kappa/2$ is a 2-uniformly convex space with some parameter $k \in]0, 2]$ satisfying the conditions (A), (B), (C).

3. L -CONVEX SPACES OF BUSEMANN TYPE

Definition 3.1 (*L -Convexity of Busemann Type*, cf. Ohta [28]). Let $L_1, L_2 \geq 0$. A metric space (Y, d) is called an *L -convex space for (L_1, L_2) in the sense of Busemann* if (Y, d) is a geodesic space, and for any three points $x, y, z \in Y$ and any minimal geodesics $\gamma := \gamma_{xy} : [0, 1] \rightarrow Y$ and $\eta := \gamma_{xz} : [0, 1] \rightarrow Y$, and for all $t \in [0, 1]$,

$$(3.1) \quad d(\gamma_t, \eta_t) \leq \left(1 + L_1 \frac{\min\{d(x, y) + d(x, z), 2L_2\}}{2} \right) td(y, z)$$

holds. A metric space (Y, d) is called a *weakly L -convex space for (L_1, L_2) in the sense of Busemann* if (Y, d) is a geodesic space, and for any three points $x, y, z \in Y$ and any minimal geodesics $\gamma := \gamma_{xy} : [0, 1] \rightarrow Y$ and $\eta := \gamma_{xz} : [0, 1] \rightarrow Y$, and for all $t \in [0, 1]$,

$$(3.2) \quad d(\gamma_t, \eta_t) \leq (1 + L_1 L_2) td(y, z)$$

holds. A metric space (Y, d) is said to be *quasi- L -convex for (L_1, L_2) in the sense of Busemann* if (Y, d) is weakly L -convex for (L_1, L_2) in the sense of Busemann such that for any $x \in Y$, any two minimal geodesics γ and η emanating from x and $t, s \in [0, \infty[$, the limit

$$(3.3) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d(\gamma_{t\varepsilon}, \eta_{s\varepsilon})$$

always exists.

Clearly, any complete separable $\text{CAT}(0)$ -space is an L -convex space for (L_1, L_2) with $L_1 L_2 = 0$ in the sense of Busemann. Let (Y, d) be a $\text{CAT}(1)$ -space with $\text{diam}(Y) \leq \pi - \varepsilon$, $\varepsilon \in]0, \pi[$ in which no triangle has a perimeter greater than 2π . Then by Proposition 4.1 in [28], (Y, d) is an L -convex space for

$$(L_1, L_2) = \left(\frac{2\{(\pi - \varepsilon) - \sin \varepsilon\}}{(\pi - \varepsilon) \sin \varepsilon}, \pi - \varepsilon \right).$$

By Lemma 4.1 in [28], L -convexity of Busemann type implies the quasi- L -convexity of Busemann type.

Let (Y, d) be a quasi- L -convex space for some (L_1, L_2) . For $x \in Y$, we define Σ'_x as the set of unit speed minimal geodesics emanating from $x \in Y$. Then $\gamma, \eta \in \Sigma'_x$ and $t, s \in [0, \infty[$, we can define the limit $\lim_{\varepsilon \rightarrow 0} d(\gamma_{t\varepsilon}, \eta_{s\varepsilon})/\varepsilon$. Define the *space of directions* Σ_x at $x \in X$ by $\Sigma_x := \Sigma'_x / \sim$, where $\gamma \sim \eta$ holds if $\lim_{\varepsilon \rightarrow 0} d(\gamma_\varepsilon, \eta_\varepsilon)/\varepsilon = 0$. Put

$$K'_x := \Sigma_x \times [0, \infty[/ \sim,$$

where $(\gamma, t) \sim (\eta, s)$ holds if $\lim_{\varepsilon \rightarrow 0} d(\gamma_{t\varepsilon}, \eta_{s\varepsilon})/\varepsilon = 0$. Then

$$d_{K'_x}((\gamma, t), (\eta, s)) := \lim_{\varepsilon \rightarrow 0} \frac{d(\gamma_{t\varepsilon}, \eta_{s\varepsilon})}{\varepsilon}$$

gives a distance function on K'_x . Define the *tangent cone* (K_x, d_{K_x}) at $x \in X$ as the completion of $(K'_x, d_{K'_x})$.

The following proposition can be similarly proved as for Proposition 4.2 in [28].

Proposition 3.2 (cf. Proposition 4.2 in [28]). *For a p -uniformly convex space (Y, d) having the quasi- L -convexity of Busemann type for some (L_1, L_2) and $x \in Y$, the tangent cone (K_x, d_{K_x}) is a geodesic space. Moreover, it is weakly L -convex in the sense of Busemann with $L_1 L_2 = 0$, that is, a Busemann's NPC space.*

4. WEAK CONVERGENCE OVER p -UNIFORMLY CONVEX SPACES

Throughout this section, we denote by i any element of a given directed set $\{i\}$. We need the following:

Proposition 4.1. *Let $\{(H_i, d_{H_i})\}$ be a net of complete p -uniformly convex spaces with common parameter $k \in]0, 2]$ and all (H_i, d_{H_i}) have the weak L -convexity of Busemann type for some common (L_1, L_2) . Let $x_i \in H_i$ be a net and $\gamma^i, \eta^i : [0, 1] \rightarrow H_i$ a net of minimal segments. Set*

$$\alpha_0 := \overline{\lim}_i d_{H_i}(\gamma_0^i, \eta_0^i), \quad \alpha_1 := \overline{\lim}_i d_{H_i}(\gamma_1^i, \eta_1^i)$$

and $A := (1 + L_1 L_2)(\alpha_0 + \alpha_1)$. Then

$$\overline{\lim}_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

or

$$\overline{\lim}_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

holds.

Corollary 4.2. *Let $\{(H_i, d_{H_i})\}$ be a net of complete p -uniformly convex spaces with common parameter $k \in]0, 2]$ and all (H_i, d_{H_i}) have the weak*

L -convexity of Busemann type for some common (L_1, L_2) . Let $x_i \in H_i$ be a net and $\gamma^i, \eta^i : [0, 1] \rightarrow H_i$ a net of minimal segments. If

$$\lim_i d_{H_i}(\gamma_0^i, \eta_0^i) = \lim_i d_{H_i}(\gamma_1^i, \eta_1^i) = 0$$

holds, then

$$\lim_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) = 0.$$

Let $\{(H_i, d_{H_i})\}$ be a net of metric spaces and (H, d_H) a metric space. Define

$$\mathcal{H} := \left(\bigsqcup_i H_i \right) \sqcup H \quad (\text{disjoint union}).$$

Definition 4.3 (Asymptotic Relation on \mathcal{H}). We call a topology on \mathcal{H} satisfying the following (A1)–(A4) an *asymptotic relation between* $\{(H_i, d_{H_i})\}$ and (H, d_H) .

- (A1) H_i and H are all closed in \mathcal{H} and the restricted topology of \mathcal{H} on each of H_i and H coincides with its original topology.
- (A2) For any $x \in H$ there exists a net $x_i \in H_i$ converging to x in \mathcal{H} .
- (A3) If $H_i \ni x_i \rightarrow x \in H$ and $H_i \ni y_i \rightarrow y \in H$ in \mathcal{H} , then we have $d_{H_i}(x_i, y_i) \rightarrow d_H(x, y)$.
- (A4) If $H_i \ni x_i \rightarrow x \in H$ in \mathcal{H} and if $y_i \in H_i$ is a net with $d_{H_i}(x_i, y_i) \rightarrow 0$, then $y_i \rightarrow x$ in \mathcal{H} .

Definition 4.4 (Asymptotic Compactness of Asymptotic Relation). Assume that $\{(H_i, d_{H_i})\}$ and (H, d_H) have an asymptotic relation. We say that a net $x_i \in H_i$ is *bounded* if $d_{H_i}(x_i, o_i)$ is bounded for some convergent net $o_i \in H_i$. The asymptotic relation is said to be *asymptotically compact* if any bounded net $x_i \in H_i$ has a convergent subnet in \mathcal{H} with respect to the asymptotic relation.

Hereafter, strong convergence on \mathcal{H} means the convergence with respect to a given asymptotic relation over \mathcal{H} . Assume that an asymptotic relation between metric spaces $\{H_i\}$ and H given. Consider the following condition:

- (G) If $\gamma^i : [0, 1] \rightarrow H_i$ and $\gamma : [0, 1] \rightarrow H$ are minimal geodesics such that $\gamma_0^i \rightarrow \gamma_0$ and $\gamma_1^i \rightarrow \gamma_1$, then $\gamma_t^i \rightarrow \gamma_t$ for any $t \in [0, 1]$.

Proposition 4.5. (1) If (G) is satisfied and if each H_i is a geodesic space, then H is so.

- (2) If (G) is satisfied and if each H_i is p -uniformly convex with common parameter $k \in]0, 2]$, then H is so.
- (3) If each H_i is p -uniformly convex with common parameter $k \in]0, 2]$ and H is a geodesic space, then (G) is satisfied and H is p -uniformly convex with parameter $k \in]0, 2]$.

In the proof of Proposition 4.5, we use Proposition 2.6.

We now define the *weak convergence* over \mathcal{H} , which generalize the notions introduced in [8, 6, 20].

Definition 4.6 (Weak Convergence on \mathcal{H}). Let $\{(H_i, d_{H_i})\}$ be a net of complete p -uniformly convex spaces with common parameter $k \in]0, 2]$ and (H, d_H) a complete p -uniformly convex space with the same parameter k . We say that a net $x_i \in H_i$ *weakly converges* to a point $x \in H$ if for any net of geodesic segments γ^i in H_i strongly converging to a geodesic segment γ in H with $\gamma_0 = x$, $\pi_{\gamma^i}(x_i)$ strongly converges to x . Here the strong convergence of $\{\gamma^i\}$ to γ means that for any $t \in [0, 1]$, γ_t^i strongly converges to γ_t . It is easy to prove that a strong convergence implies a weak convergence and that a weakly convergent net always has a unique weak limit.

The following proposition is omitted in [20]. We shall give it for completeness.

Proposition 4.7 (Weak Topology on \mathcal{H}). *The weak convergence over \mathcal{H} of complete p -uniformly convex spaces with parameter $k \in]0, 2]$ induces a Hausdorff topology on it. We call it weak topology of (H, d_H) .*

Remark 4.8. The notion of weak convergence over a fixed CAT(0)-space is proposed by Jost [8]. In [20], we extend it over \mathcal{H} of CAT(0)-spaces. In Kirk-Panyanak [14], they give a different approach on the weak convergence, so-called Δ -convergence, and Espínola and Fernández-León [6] proved the equivalence between the weak convergence and the Δ -convergence over a fixed CAT(0)-space or CAT(1)-space whose diameter strictly less than $\pi/2$ (see Proposition 5.2 in [6]). Such an equivalence is also valid for a fixed p -uniformly convex space in the same way as in the proof of Proposition 5.2 in [6].

Lemma 4.9. *Let $\{(H_i, d_{H_i})\}$ be a net of complete p -uniformly convex space with common parameter $k \in]0, 2]$ and (H, d_H) a complete p -uniformly convex space with the same parameter k . Suppose that a net $x_i \in H_i$ is weakly convergent to $x \in H$ and a net $y_i \in H_i$ is strongly convergent to $y \in H$. Then we have the following:*

- (1) *Under (A) for all (H_i, d_{H_i}) , $d_H(x, y) \leq \liminf_i d_{H_i}(x_i, y_i)$.*
- (2) *Under (B) for all (H_i, d_{H_i}) , $\lim_i d_{H_i}(x_i, y_i) = d_H(x, y)$ if and only if $x_i \in H_i$ strongly converges to $x \in H$.*

The main result of this section is the following theorem:

Theorem 4.10 (Banach-Alaoglu Type Theorem). *Let $\{(H_i, d_{H_i})\}$ be a net of complete p -uniformly convex spaces with common parameter $k \in]0, 2]$ and (H, d_H) a complete p -uniformly convex space with the same parameter k and all (H_i, d_{H_i}) and (H, d_H) have the weak L -convexity of Busemann type for some common (L_1, L_2) . Suppose one of the following:*

- (1) *(B) and (C) hold for (H, d_H) and $(H_i, d_{H_i}) = (H, d_H)$ holds for all i .*
- (2) *(H, d_H) is separable.*

Then every bounded net $\{x_i\} \subset \mathcal{H}$ has a weakly convergent subsequence.

Combining Theorems 2.13 and 4.10, we obtain the following:

Corollary 4.11 (Banach-Alaoglu Type Theorem over $\text{CAT}(\kappa)$ -Spaces). *Let $\{(H_i, d_{H_i})\}$ be a net of complete $\text{CAT}(\kappa)$ -spaces with $\text{diam}(H_i) < R_\kappa/2 - \varepsilon$ with $\varepsilon \in]0, R_\kappa/2[$, and (H, d_H) a complete $\text{CAT}(\kappa)$ -space with $\text{diam}(H) < R_\kappa/2 - \varepsilon$ with $\varepsilon \in]0, R_\kappa/2[$. Assume that $(H_i, d_{H_i}) = (H, d_H)$ for all i or (H, d_H) is separable. Then every bounded net $\{x_i\} \subset \mathcal{H}$ has a weakly convergent subsequence.*

Remark 4.12. The assertion of Theorem 4.10 was proved by Theorem 2.1 in Jost [8] over a fixed complete $\text{CAT}(0)$ -space without assuming the separability. In the framework of convergence over $\text{CAT}(0)$ -spaces, Lemma 5.5 in [20] extends Theorem 2.1 in [8]. For a fixed $\text{CAT}(\kappa)$ -space (H, d_H) with $\text{diam}(H) < R_\kappa/2 - \varepsilon$ with $\varepsilon \in]0, R_\kappa/2[$, the assertion of Corollary 4.11 is essentially shown by combining Corollary 4.4 and Remark 5.3 of [6]. Corollary 4.11 also extends the result in [6].

5. VARIATIONAL CONVERGENCE OVER p -UNIFORMLY CONVEX SPACES

In this section we fix $p \geq 2$.

5.1. Resolvents. Throughout this subsection, we fix a complete p -uniformly convex space (H, d_H) with parameter $k \in]0, 2]$. Consider a function $E : H \rightarrow [0, \infty]$ and set $D(E) := \{x \in H \mid E(x) < \infty\}$.

Definition 5.1 (Moreau-Yosida Approximation, [9]). For $E : H \rightarrow [0, +\infty]$ we define $E^\lambda : H \rightarrow [0, +\infty]$ by

$$E^\lambda(x) := \inf_{y \in H} (\lambda^{p-1} E(y) + d_H^p(y, x)), \quad x \in H, \lambda > 0,$$

and call it the *Moreau-Yosida approximation* or the *Hopf-Lax formula* for E .

Theorem 5.2 (Existence of Resolvent). *If E is lower semi-continuous, convex and $E \not\equiv +\infty$, then for any $x \in H$ there exists a unique point, say $J_\lambda(x) \in H$, such that*

$$E^\lambda(x) = \lambda^{p-1} E(J_\lambda(x)) + d_H^p(x, J_\lambda(x)).$$

This defines a map $J_\lambda : H \rightarrow H$, called the resolvent of E .

Note that if H is a Hilbert space and $p = 2$, and if E is a closed densely defined non-negative quadratic form on H , then we have $J_\lambda = (I + \lambda A)^{-1} = \frac{1}{\lambda} G_{\frac{1}{\lambda}}$. Here, I is the identity operator, A the infinitesimal generator associated with E , i.e., the non-negative self-adjoint operator on H such that $D(E) = \sqrt{A}$ and $E(x) = (\sqrt{A}x, \sqrt{A}x)_H$ for any $x \in D(E)$, where $(\cdot, \cdot)_H$ is the Hilbert inner product on H , and $G_\alpha = (\alpha + A)^{-1}$, $\alpha > 0$ is the resolvent operator associated with A .

To the end of this subsection, we always assume the convexity of E . We have the following lemmas and theorems which are known for the case that (H, d_H) is a CAT(0)-space. The proofs are omitted.

Lemma 5.3. *For $\lambda, \mu > 0$, we have*

$$\frac{1}{\mu^{p-1}} \left(\frac{1}{\lambda^{p-1}} E^\lambda \right)^\mu = \frac{1}{(\lambda + \mu)^{p-1}} E^{\lambda+\mu}.$$

Lemma 5.4. *Let $E : H \rightarrow [0, \infty]$ be a lower semi-continuous function with $E \not\equiv \infty$. For $x \in H$ and $s \in [0, 1]$, we have*

$$J_\lambda(x) = J_{(1-s)\lambda}((1-s)x + sJ_\lambda(x)),$$

where $(1-s)x + sJ_\lambda(x)$ is the point on the geodesic joining x to $J_\lambda(x)$ such that $d_H(x, (1-s)x + sJ_\lambda(x)) = sd_H(x, J_\lambda(x))$.

Lemma 5.5. *Let $J_\lambda : H \rightarrow H$, $\lambda > 0$ be the resolvent associated with a lower semi-continuous convex function $E : H \rightarrow [0, \infty]$ with $E \not\equiv \infty$. For $x \in \overline{D(E)}$, then*

$$\lim_{\lambda \rightarrow 0} d_H(J_\lambda(x), x) = 0.$$

Theorem 5.6. *Let $E : H \rightarrow [0, \infty]$ be a lower semi-continuous convex function with $E \not\equiv \infty$. Take $x \in H$ and assume that $(J_{\lambda_n}(x))_{n \in \mathbb{N}}$ is bounded for some sequence $\lambda_n \rightarrow \infty$. Then $(J_\lambda(x))_{\lambda > 0}$ converges to a minimizer of E .*

5.2. Variational Convergence. Throughout this subsection, we fix a net $\{(H_i, d_{H_i})\}$ of complete p -uniformly convex spaces with common parameter $k \in]0, 2]$ and a complete p -uniformly convex space (H, d) with the same parameter $k \in]0, 2]$. Consider a net $\{E_i\}$ of functions $E_i : H_i \rightarrow [0, \infty]$ and a function $E : H \rightarrow [0, \infty]$.

Definition 5.7 (Asymptotic Compactness, [24],[20]). The net $\{E_i\}$ of functions is said to be *asymptotically compact* if for any bounded net $x_i \in H$ with $\lim_i E_i(x_i) < +\infty$ there exists a convergent subnet of $\{x_i\}$.

Definition 5.8 (Γ -convergence). We say that E_i Γ -converges to E if the following ($\Gamma 1$) and ($\Gamma 2$) are satisfied:

($\Gamma 1$) For any $x \in H$ there exists a net $x_i \in H_i$ such that $x_i \rightarrow x$ and $E_i(x_i) \rightarrow E(x)$.

($\Gamma 2$) If $H_i \ni x_i \rightarrow x \in H$ then $E(x) \leq \underline{\lim}_i E_i(x_i)$.

Definition 5.9 (Mosco convergence). We say that E_i converges to E in the Mosco sense if both ($\Gamma 1$) in Definition 5.8 and the following ($\Gamma 2'$) hold.

($\Gamma 2'$) If $H_i \ni x_i \rightarrow x \in H$ weakly, then $E(x) \leq \underline{\lim}_i E_i(x_i)$.

Note that ($\Gamma 2'$) is a stronger condition than ($\Gamma 2$), so that a Mosco convergence implies a Γ -convergence.

It is easy to prove the following proposition. The proof is omitted.

Proposition 5.10. *Assume that $\{E_i\}$ is asymptotically compact. Then the following (1)–(3) are all equivalent to each other.*

- (1) E_i converges to E in the Mosco sense.
- (2) E_i Γ -converges to E .
- (3) E_i compactly converges to E .

In what follows, we assume that all H_i and H are p -uniformly convex spaces with a common parameter $k \in]0, 2]$ having the weak L -convexity of Busemann type, and all functions $E_i : H_i \rightarrow [0, +\infty]$ and $E : H \rightarrow [0, +\infty]$ are all *lower semi-continuous, convex, and are not identically equal to $+\infty$* . Let J_λ^i and J_λ be the resolvents of E_i and E respectively.

Theorem 5.11. *Suppose that all (H_i, d_{H_i}) satisfy the condition (B). Assume that $(H_i, d_{H_i}) = (H, d_H)$ for all i and (H, d_H) satisfies (C), or (H, d_H) is separable. If E_i converges to E in the Mosco sense, then for any $\lambda > 0$ we have the following (1) and (2).*

- (1) E_i^λ strongly converges to E^λ .
- (2) J_λ^i strongly converges to J_λ .

Proposition 5.12. *If E_i^λ strongly converges to E^λ for any $\lambda > 0$, then E_i Γ -converges to E .*

Propositions 5.10, 5.12 and Theorem 5.11 together imply the following

Corollary 5.13. *Assume that $\{E_i\}$ is asymptotically compact and all (H_i, d_{H_i}) satisfies the condition (A). Then, the following (1) and (2) are equivalent.*

- (1) E_i compactly converges to E .
- (2) E_i^λ strongly converges to E^λ for any $\lambda > 0$.

6. CHEEGER TYPE SOBOLEV SPACE OVER L^p -MAPS

In this section, we prepare several notions for our main Theorem 1.1.

6.1. The space of L^p -maps. Let (X, \mathcal{X}, m) be a σ -finite measure space. Denote by \mathcal{X}^m the completion of \mathcal{X} with respect to m . In what follows, we simply say *measurable* (resp. \mathcal{X}^m -*measurable*) for \mathcal{X} -measurable (resp. \mathcal{X}^m -measurable). A numerical function f on X is a map $f : X \rightarrow [-\infty, \infty]$. For a measurable numerical function f on X , we set $\|f\|_p := (\int_X |f(x)|^p m(dx))^{1/p}$, $\|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \text{ } m\text{-a.e. } x \in X\}$. For $p \in]0, \infty]$, denote by $L^p(X; m)$ the family of m -equivalence classes of \mathcal{X}^m -measurable functions finite with respect to $\|\cdot\|_p$. Denote by $L^0(X; m)$ the family of m -equivalence classes of \mathcal{X}^m -measurable numerical functions $f : X \rightarrow [-\infty, \infty]$ with $|f| < \infty$ m -a.e.

Let (Y, d) be a metric space. For $p \in]0, \infty]$ and measurable maps $f, g : X \rightarrow Y$, define a pseudo distance $d_p(f, g)$ by $d_p(f, g) := \|d(f, g)\|_p$. If $p < \infty$, then

$$d_p(f, g) := \left(\int_X d^p(f(x), g(x)) m(dx) \right)^{1/p}.$$

If $p = \infty$, then $d_\infty(f, g)$ is the m -essentially supremum of $x \mapsto d(f(x), g(x))$. We say that f and g are m -equivalent if

$$f(x) = g(x) \quad m\text{-a.e. } x \in X$$

and write $f \stackrel{m}{\sim} g$. For a fixed measurable map $h : X \rightarrow Y$, we set

$$L_h^p(X, Y; m) := \{f \in \mathcal{X}/\mathcal{B}(Y) \mid d(f, h) \in L^p(X; m)\} / \stackrel{m}{\sim}.$$

The map $h : X \rightarrow Y$ is called a *base map*. If $m(X) < \infty$ and $h : X \rightarrow Y$ is bounded, then $L_h^p(X, Y; m)$ is independent of the choice of such h .

Lemma 6.1. *Let (Y, d) be a metric space. For a fixed measurable map $h : X \rightarrow Y$ and $p \in [1, \infty]$, we have the following:*

- (1) *If (Y, d) is complete (resp. separable), then $(L_h^p(X, Y; m), d_p)$ is so.*
- (2) *Suppose that (Y, d) is a geodesic space and any two points can be connected by a unique minimal geodesic. For given $\gamma_0, \gamma_1 \in Y$ and each $t \in [0, 1]$, let γ_t be the t -point in a unique minimal geodesic $\gamma : [0, 1] \rightarrow Y$ joining γ_0 to γ_1 . Assume that for each $t \in [0, 1]$, γ_t is continuous with respect to (γ_0, γ_1) . Then for given $f_0, f_1 \in L_h^p(X, Y; m)$, the map $f_t : X \rightarrow Y$ defined by $f_t(x) := (f_0(x)f_1(x))_t$ belongs to $L_h^p(X, Y; m)$ and forms a minimal geodesic joining f_0 to f_1 in $L_h^p(X, Y; m)$. In particular, $(L_h^p(X, Y; m), d_p)$ is a geodesic space.*

Theorem 6.2. *Let (Y, d) be a complete p -uniformly convex space having the weak L -convexity of Busemann type. Fix a measurable map $h : X \rightarrow Y$. Then we have the following:*

- (1) *$(L_h^p(X, Y; m), d_p)$ is a complete p -uniformly convex space having the weak L -convexity of Busemann type.*
- (2) *Let $\gamma : [0, \infty[\rightarrow L_h^p(X, Y; m)$ be a minimal geodesic. Then for each $x \in X$ and $L \in [0, \infty[$, there exists a minimal segment $\tilde{\gamma}^{(L)}(x) : [0, L] \rightarrow Y$ such that $d_p(\gamma_t, \tilde{\gamma}_t^{(L)}) = 0$ for all $t \in [0, L]$, where $\tilde{\gamma}^{(L)} : [0, L] \rightarrow L_h^p(X, Y; m)$ is a minimal segment defined by $\tilde{\gamma}^{(L)}(x)$.*
- (3) *Assume that (Y, d) satisfies the quasi- L -convexity of Busemann type for some (L_1, L_2) . Then $(L_h^p(X, Y; m), d_p)$ is so.*

Lemma 6.3. *Let (Y, d) be a complete p -uniformly convex space having the weak L -convexity of Busemann type such that (Y, d) satisfies (A). Let F be a closed convex subset of $(L_h^p(X, Y; m), d_p)$. For each $x \in X$, set $F(x) := \{f(x) \mid f \in F\}$.*

- (1) For each $x \in X$, $F(x)$ is convex in (Y, d) .
- (2) Take an $f \in L_h^p(X, Y; m)$. Then $\pi_F(f) = (\pi_{\overline{F(x)}}(f(x)))_{x \in X}$ in $L_h^p(X, Y; m)$.

Theorem 6.4. *Let (Y, d) be a complete p -uniformly convex space having the weak L -convexity of Busemann type. The following hold:*

- (1) *If (Y, d) satisfies (A), then $(L_h^p(X, Y; m), d_p)$ does so.*
- (2) *If (Y, d) satisfies (B), then $(L_h^p(X, Y; m), d_p)$ does so.*
- (3) *If (Y, d) satisfies (C), then $(L_h^p(X, Y; m), d_p)$ does so.*

Corollary 6.5. *For $p \geq 2$, $L^p(X; m)$ satisfies (A), (B), (C).*

Corollary 6.6. *Let (Y, d) be a complete $CAT(\kappa)$ -space with a diameter strictly less than $R_\kappa/2$. Then we have the following:*

- (1) *$(L_h^2(X, Y; m), d_2)$ is a 2-uniformly convex space with the same parameter $k \in]0, 2]$ having the weak L -convexity of Busemann type.*
- (2) *$(L_h^2(X, Y; m), d_2)$ satisfies (A), (B) and (C).*

Hereafter, we focus only on the case that X is a locally compact separable metric space and $h \equiv o$, where $o \in Y$ is a fixed base point. We write $L_o^r(X, Y; m)$ instead of $L_h^r(X, Y; m)$ in such a case.

Definition 6.7 (Lipschitz Maps with Compact Support). The *support* ‘ $\text{supp}[u]$ ’ for a measurable map $u : X \rightarrow Y$ is defined to be the subset of X satisfying the condition that $x \in X \setminus \text{supp}[u]$ if and only if there exists an open neighborhood U of x such that $u = o$ on U . Denote by $C_o^{\text{Lip}}(X, Y)$ the set of Lipschitz continuous maps $u : X \rightarrow Y$ with compact support $\text{supp}[u]$.

Theorem 6.8. *Suppose that (Y, d) is a separable geodesic space. Let $r \geq 1$. Then $C_o^{\text{Lip}}(X, Y)$ is a dense subset of $(L_o^r(X, Y; m), d_r)$.*

6.2. Upper gradient and Cheeger’s Sobolev spaces. In what follows, let (X, d_X) be a metric space, and $U \subset X$ be an open set, and m be a Borel regular measure on X such that any ball with finite positive radius is of finite positive measure. Let (Y, d) be a complete geodesic space.

Definition 6.9 (Upper Gradient). A Borel function $g : U \rightarrow [0, \infty]$ is called an *upper gradient* for a map $u : U \rightarrow Y$ if, for any unit speed curve $c : [0, \ell] \rightarrow U$, we have

$$\Phi(u(c(0)), u(c(\ell))) \leq \int_0^\ell g(c(s)) ds.$$

Definition 6.10 (Upper Pointwise Lipschitz Constant Function). For a map $u : U \rightarrow Y$ and a point $z \in U$, we define

$$\begin{aligned} \text{Lip } u(z) &:= \lim_{r \rightarrow 0} \sup_{d_X(z, w) = r} \frac{d(u(z), u(w))}{r}, \\ \text{Lip } u(z) &:= \lim_{r \rightarrow 0} \sup_{0 < d_X(z, w) < r} \frac{d(u(z), u(w))}{d_X(z, w)} \end{aligned}$$

and we put $\text{Lip } u(z) = \text{Lip } u(z) = 0$ if z is an isolated point. Clearly $\text{Lip } u \leq \text{Lip } u$ on X . We call $\text{Lip } u$ the *upper pointwise Lipschitz constant function* for u .

Cheeger [4] proved that for a locally Lipschitz function $u : U \rightarrow \mathbb{R}$, then $\text{Lip } u$, hence $\text{Lip } u$, is an upper gradient for u . We next define the Cheeger type Sobolev spaces. Fix a point $o \in Y$ as a base point and $p \in [1, \infty[$. Let $L_o^p(U, Y; m)$ be the space of L^p -maps as defined in the previous section. We write $L^p(U, Y; m)$ instead of $L_o^p(U, Y; m)$ for simplicity.

Definition 6.11 (Cheeger Type Sobolev Space). For $u \in L^p(U, Y; m)$, we define the Cheeger type p -energy of u as

$$E_p(u) := \inf_{\{(u_i, g_i)\}_{i=1}^\infty} \lim_{i \rightarrow \infty} \|g_i\|_{L^p(U; m)}^p,$$

where the infimum is taken over all sequences $\{(u_i, g_i)\}_{i=1}^\infty$ such that $u_i \rightarrow u$ in $L^p(U, Y; m)$ as $i \rightarrow \infty$ and g_i is an upper gradient for u_i for each i . The *Cheeger type $(1, p)$ -Sobolev space* is defined by

$$H^{1,p}(U, Y; m) := \{u \in L^p(U, Y; m) \mid E_p(u) < \infty\}.$$

By definition, if $u = v$ m -a.e. on U , then $E_p(u) = E_p(v)$.

The following is proved in [26].

Theorem 6.12 (Lower Semi Continuity of Energy, see Theorem 2.8 in [26]). *If a sequence $\{u_i\}_{i=1}^\infty$ converges to u in $L^p(U, Y; m)$, then $E_p(u) \leq \liminf_{i \rightarrow \infty} E_p(u_i)$.*

Definition 6.13 (Generalized Upper Gradient). A function $g \in L^p(U; m)$ is called a *generalized upper gradient* for $u \in H^{1,p}(U, Y; m)$ if there exists a sequence $\{(u_i, g_i)\}_{i=1}^\infty$ such that g_i is an upper gradient for u_i and $u_i \rightarrow u$, $g_i \rightarrow g$ in $L^p(U, Y; m)$, $L^p(U; m)$ respectively as $i \rightarrow \infty$.

From the definition of the p -energy, $E_p(u) \leq \|g\|_{L^p(U; m)}^p$ for any generalized upper gradient $g \in L^p(U; m)$ for $u \in H^{1,p}(U, Y; m)$.

Definition 6.14 (Minimal Generalized Upper Gradient). A generalized upper gradient $g \in L^p(U; m)$ for a map $u \in H^{1,p}(U, Y; m)$ is said to be *minimal* if it satisfies $E_p(u) = \|g\|_{L^p(U; m)}^p$.

Hereafter, we assume that (Y, d) is weakly L -convex with $L_1 L_2 = 0$, that is, (Y, d) is a Busemann's NPC space. Then the distance function $d : Y \times Y \rightarrow [0, \infty[$ is convex. We know the following results:

Lemma 6.15 (See, Lemma 3.1 in [28]). *Suppose that (Y, d) is weakly L -convex with $L_1 L_2 = 0$. Let $u_1, u_2 : U \rightarrow Y$ be maps. For any upper gradient g_1, g_2 for u_1, u_2 respectively and $0 \leq \lambda \leq 1$. The function $g := (1-\lambda)g_1 + \lambda g_2$ is an upper gradient for the map $v := (1-\lambda)u_1 + \lambda u_2$. In particular, for any $u_1, u_2 \in H^{1,p}(U, Y; m)$ with $1 \leq p < \infty$ and for any $0 \leq \lambda \leq 1$, we have*

$$E_p((1-\lambda)u_1 + \lambda u_2)^{1/p} \leq (1-\lambda)E_p(u_1)^{1/p} + \lambda E_p(u_2)^{1/p}.$$

Theorem 6.16 (See, Theorem 3.2 in [26]). *Let $p \in]1, \infty[$. Suppose that (Y, d) is weakly L -convex with $L_1 L_2 = 0$. Then for any $u \in H^{1,p}(U, Y; m)$, there exists a unique minimal generalized upper gradient g_u for u .*

For $p \in]1, \infty[$, we define a distance $d_{H^{1,p}}$ on $H^{1,p}(U, Y; m)$: for $u, v \in H^{1,p}(U, Y; m)$,

$$(6.1) \quad d_{H^{1,p}}(u, v) := d_p(u, v) + \|g_u - g_v\|_{L^p(U; m)},$$

where g_u, g_v is the minimal generalized upper gradient for $u, v \in H^{1,p}(U, Y; m)$, respectively. Let $(\overline{H}^{1,p}(U, Y; m), d_{\overline{H}^{1,p}})$ be the completion of $(H^{1,p}(U, Y; m), d_{H^{1,p}})$.

The following assertion is not declared clearly in [26]. We provide its proof for completeness.

Theorem 6.17. *Let $p \in]1, \infty[$. We have $\overline{H}^{1,p}(U, Y; m) = H^{1,p}(U, Y; m)$.*

Remark 6.18. Theorem 6.17 does not necessarily imply the $d_{H^{1,p}}$ -completeness of $H^{1,p}(U, Y; m)$, that is, $d_{\overline{H}^{1,p}} = d_{H^{1,p}}$ on $H^{1,p}(U, Y; m)$.

6.3. p -harmonic maps. In this subsection, we still assume that (Y, d) is weakly L -convex with $L_1 L_2 = 0$.

Definition 6.19 (p -Harmonic Map). For $v \in H^{1,p}(U, Y; m)$, let $H_v^{1,p}(U, Y; m)$ be the $d_{H^{1,p}}$ -closure of

$$\{u \in H^{1,p}(U, Y; m) \mid \text{supp } d(u, v) \Subset U\}.$$

v is said to be p -harmonic if and only if $E_p(v) = \inf_{u \in H_v^{1,p}(U, Y; m)} E_p(u)$.

Theorem 6.20. *Suppose $p \geq 2$. If there exists $C > 0$ such that for any $f \in H_0^{1,p}(U)$,*

$$\int_U |f|^p dm \leq C \int_U |g_f|^p dm, \quad (\text{Poincaré Inequality})$$

then there exists a p -harmonic map in $H_v^{1,p}(U, Y; m)$ for given $v \in H^{1,p}(U, Y; m)$.

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DEPARTMENT OF MATHEMATICS AND ENGINEERING
 GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
 KUMAMOTO UNIVERSITY
 KUMAMOTO, 860-8555
 JAPAN
E-mail address: kuwae@gpo.kumamoto-u.ac.jp